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A transport-equation description of nonlinear ocean surface wave interactions

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The evolution of the power spectrum of surface gravity waves is described by means of an energy transport equation. A slowly varying, prescribed ocean current and wind source are assumed to account for spatial inhomogeneities in the surface wave spectrum. These inhomogeneities lead to a new nonlinear wave–wave interaction mechanism.

1. Introduction

The complex turbulent structure of ocean surface waves has led to a variety of statistical descriptions of their properties. In particular, the wavenumber spectrum $\Psi(\mathbf{k})$ has provided a useful representation of ocean waves (see, for example, Phillips 1966). To describe the evolution of this spectrum a number of authors (Hasselmann 1962, 1963; Snyder & Cox 1966; Barnett 1968; Thomson & West 1973) have introduced transport equations similar to that of radiative transfer or to the Boltzmann equation of kinetic theory.

Such transport equations express the rate of change of the spectrum as a sum of terms which individually model physical phenomena thought to be important in the development of ocean waves. Of particular significance are wind generation of waves (Phillips 1957; Miles 1957, 1960), nonlinear wave–wave interactions (Phillips 1960; Hasselmann 1960, 1962, 1963; West, Watson & Thomson 1974), the effect of an imposed surface current (Whitham 1961; Kenyon 1971) and the effect of wave breaking (Thomson & West 1973). A review of the modelling of these phenomena has been given by Hasselmann (1968).

In his study of nonlinear wave–wave interactions, Hasselmann (1962, 1963) assumed the spectrum to be spatially homogeneous. The resulting contribution to the transport equation is a term of *third order* in the spectral function.

The purpose of the present paper is to study nonlinear wave–wave interactions when the spectrum varies with the position $\mathbf{x}$ on the ocean surface. New terms contributing to the transport equation are found which are of *second order* in the spectral function and which are non-vanishing only when the wave spectrum has a spatial variation. A non-uniform appearance of the sea surface is, in fact, not at all unusual. There are many possible causes of this, some of which are atmospheric wind turbulence, tidal current rips, internal wave activity, currents at the

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mouth of a bay or estuary and interaction of wind waves with a long wavelength swell. To take account of such phenomena, we shall assume that a prescribed current $\mathbf{U}(\mathbf{x}, t)$ is present which varies over distances which are large compared with the surface wavelengths of interest and which varies slowly over times comparable to the corresponding surface wave periods. A corresponding wave source will be modelled. As the wave system responds to these driving mechanisms, the nonlinear wave–wave interactions provide a coupling of the excitation across the spectrum.

2. The coupled mode equation for surface waves

In this section we shall describe surface wave dynamics with a set of nonlinear equations which couple the eigenmode amplitudes of linear waves. Similar equations have been previously developed by other authors (see, for example, Phillips 1960; Benney 1962; Hasselmann 1962, 1963). Since the various developments differ in detail, we briefly outline the derivation of our equations in appendix A.

Incompressible irrotational flow is assumed. The fluid velocity is thus expressed as the gradient of a velocity potential

$$\Phi = \phi + \hat{\Phi}. \tag{1}$$

In (1) the velocity potential is given by a linear superposition of that of the short wavelength, high frequency surface waves, represented by ϕ, and that of the long wavelength, slowly varying prescribed current, represented by $\hat{\Phi}$. The z variation of the surface current is assumed negligible for those surface waves being studied here.

The undisturbed ocean surface is assumed to coincide with the plane $z = 0$ of a rectangular co-ordinate system. The z axis is directed upwards and the x, y plane lies in this surface. The horizontal flow associated with the prescribed current is

$$\mathbf{U}(\mathbf{x}, t) = \nabla_s \hat{\Phi}, \quad z = 0, \tag{2}$$

where ∇_s is the gradient operator acting in the horizontal plane, $\mathbf{x} = (x, y)$ is a vector in this plane, and we assume $\hat{\Phi}$ to vary slowly with z so that we can evaluate $\mathbf{U}$ at $z = 0$. The effect of the vertical flow $|\partial\hat{\Phi}/\partial z|$ of the current is assumed to be negligible.†

The equation for the sea surface can therefore be expressed as

$$z = \zeta(\mathbf{x}, t), \tag{3}$$

where $\zeta(\mathbf{x}, t)$ represents the short wavelength vertical displacement due to surface gravity waves. The velocity potential at the surface is then

$$\phi_s(\mathbf{x}, t) \equiv \phi(\mathbf{x}, z, t) \quad \text{at} \quad z = \zeta(\mathbf{x}, t). \tag{4}$$

We represent the flow field by the complex amplitude $Z(\mathbf{x}, t)$, defined by the equations

$$\phi_s = \tfrac{1}{2}V_x(Z + Z^*), \quad \zeta = \tfrac{1}{2}i(Z - Z^*). \tag{5}$$

† When $\mathbf{U}$ represents the fluid motion associated with a long swell, the effect of vertical motion can be neglected when the waves being studied have a wavelength short compared with that of the swell.

Here V_x is the 'velocity operator' (g is the acceleration due to gravity):

$$V_x \equiv (g/\Theta)^{\frac{1}{2}} \equiv \omega_x/\Theta, \quad \Theta \equiv (-\nabla_s^2)^{\frac{1}{2}}. \tag{6}$$

These quantities are assumed to operate on functions expressed as Fourier series, for which the proper operation is self-evident. For example, we assume Z to be a function defined in a rectangular ocean of area A_0 (with periodic boundary conditions) and write

$$Z(\mathbf{x}, t) = \sum_{\mathbf{k}} A(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x}), \tag{7}$$

where the Fourier coefficients $A(\mathbf{k})$ are time dependent. Thus

$$V_x Z = \sum_{\mathbf{k}} V_k A(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x}),$$

where

$$V_k \equiv (g/k)^{\frac{1}{2}} \equiv \omega_k/k \tag{8}$$

is the phase velocity of a small amplitude surface gravity wave of wavenumber k in deep water. The corresponding angular frequency is $\omega_k = (gk)^{\frac{1}{2}}$.

In our rectangular two-dimensional space representing the quiescent ocean surface, the Fourier exponentials satisfy the relations

$$\begin{aligned} A_0^{-1} \int d^2x \exp(i\mathbf{k}.\mathbf{x}) &= \delta_{\mathbf{k}}, \\ A_0^{-1} \sum_{\mathbf{k}} \exp(i\mathbf{k}.\mathbf{x}) &= \delta(\mathbf{x}), \end{aligned} \tag{9}$$

where $\delta_{\mathbf{k}}$ is the Kronecker and $\delta(\mathbf{x})$ the Dirac delta function.

The prescribed current is given the Fourier representation

$$\mathbf{U}(\mathbf{x}, t) = \sum_{\mathbf{K}} \mathbf{U}(\mathbf{K}) \cos(\mathbf{K}.\mathbf{x} - \Omega_K t), \tag{10}$$

where Ω_K and $\mathbf{U}(\mathbf{K})$ are the appropriate functions of the wavenumber $\mathbf{K}$.

It is straightforward to obtain from the fluid-dynamic equations the first-order differential equation expressing the time rate of change of the $A(\mathbf{k})$. The procedure is outlined in appendix A. The resulting equation is (here $\dot{A} \equiv dA/dt$)

$$\dot{A}(\mathbf{k}) + i\omega_k A(\mathbf{k}) = T_W(A) + T_U(A) + T_2(A) + T_3(A) + \dots. \tag{11}$$

In obtaining (11) we have neglected surface tension and have supposed the ocean to be much deeper than the longest wavelengths considered. We must therefore set equal to zero those amplitudes corresponding to wavelengths comparable to or greater than the depth in (11).

The term T_W models the effect of wind and viscosity. On the basis of a model of Miles (1957, 1960) we adopt for this the simple linear expression

$$T_W(A) = \alpha(\mathbf{k}) A(\mathbf{k}). \tag{12}$$

Models for the coefficient $\alpha(\mathbf{k})$ have also been described by Phillips (1966).

The quantity T_U describes the coupling of the surface wave field to the prescribed current. From the equations in appendix A, this quantity has the form

$$T_U = -i \sum_{\mathbf{K}} [C^-(\mathbf{k}, \mathbf{K}) A(\mathbf{k} - \mathbf{K}) \exp(-i\Omega_K t) + C^+(\mathbf{k}, \mathbf{K}) A(\mathbf{k} + \mathbf{K}) \exp(i\Omega_K t)], \tag{13}$$

where

$$C^{\pm}(\mathbf{k}, \mathbf{K}) = \tfrac{1}{4}\mathbf{U}(\mathbf{K}).[\mathbf{k} + (\mathbf{k} \pm \mathbf{K})\,\omega_k/\omega_{|\mathbf{k}\pm\mathbf{K}|}]. \tag{14}$$

The term T_2 in (11) represents the nonlinear wave–wave interaction $O(A^2)$. This is derived in appendix A and has the form

$$T_2(A) = \sum_{\mathbf{l},\mathbf{p}} \delta_{\mathbf{k}-\mathbf{l}-\mathbf{p}} [\Gamma^{\mathbf{k}}_{\mathbf{l},\mathbf{p}} A(\mathbf{l})\, A(\mathbf{p}) + \Gamma^{\mathbf{k},-\mathbf{p}}_{\mathbf{l}} A(\mathbf{l})\, A^*(-\mathbf{p}) + \Gamma^{\mathbf{k},-\mathbf{p},-\mathbf{l}} A^*(-\mathbf{l})\, A^*(-\mathbf{p})]. \tag{15}$$

The explicit expressions for the coefficients Γ are given in equation (A 8) of appendix A.

Finally, the term T_3 describes nonlinear wave interactions $O(A^3)$. This is shown in appendix A to have the form

$$T_3(A) = \frac{i}{4} \sum_{\mathbf{l},\mathbf{p},\mathbf{n}} \delta_{\mathbf{k}+\mathbf{n}-\mathbf{l}-\mathbf{p}} [\Gamma^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}} A(\mathbf{l})\, A(\mathbf{p})\, A^*(\mathbf{n}) + \Gamma^{\mathbf{k}}_{\mathbf{l},\mathbf{p},-\mathbf{n}} A(\mathbf{l})\, A(\mathbf{p})\, A(-\mathbf{n})$$
$$+ \Gamma^{\mathbf{k},-\mathbf{p},\mathbf{n}}_{\mathbf{l}} A(\mathbf{l})\, A^*(-\mathbf{p})\, A^*(\mathbf{n}) + \Gamma^{\mathbf{k},-\mathbf{p},-\mathbf{l},\mathbf{n}} A^*(-\mathbf{l})\, A^*(-\mathbf{p})\, A^*(\mathbf{n})]. \tag{16}$$

Of the coefficients in (16), only $\Gamma^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}}$ will be needed in this paper. This is given in equation (A 9) of appendix A. The remaining terms in (16) tend to involve rapidly oscillating exponentials and are not expected to contribute significantly to transfer of excitation between modes, i.e. linear waves, in (11).

The dots in (11) indicate that we have neglected higher-order terms in the $A(\mathbf{k})$ and $\mathbf{U}(\mathbf{K})$. The terms kept are of the lowest order required to describe the transport phenomena of interest to us here. To obtain Hasselmann's (1962, 1963) transport equation, we should require terms to $O(A^5)$ in (11).

It will prove convenient to eliminate the term T_2 from (11). To do this we write

$$A(\mathbf{k}) = a(\mathbf{k}) + G(\mathbf{k}), \tag{17}$$

where G satisfies the equation

$$\dot{G}(\mathbf{k}) + i\omega_k G(\mathbf{k}) = T_2(a). \tag{18}$$

This equation may be formally integrated to give

$$G(\mathbf{k}) = \exp(-i\omega_k t) \int^t \exp(i\omega_k t')\, T_2(a)\, dt'. \tag{19}$$

Thus G is $O(a^2)$. The difference $T_2'(a) \equiv T_2(A) - T_2(a)$, expressed as a functional of the a's, is $O(a^3)$. This lets us finally rewrite (11) in the form

$$\dot{a}(\mathbf{k}) + i\omega_k a(\mathbf{k}) = T_W(a) + T_U(a) + T_3'(a), \tag{20}$$

where

$$T_3'(a) \equiv T_3(a) + T_2'(a) \tag{21}$$

and we have dropped terms of order higher than a^3. We have also dropped the higher-order terms in T_W and T_U, which is consistent with our use of only simple linear models for these phenomena.

We shall formally suppose that T_W, T_U and T_3 are of the same order of smallness in (20). This permits us to evaluate (19) in a simple approximation: writing $a(\mathbf{k}) = q(\mathbf{k}) \exp(-i\omega_k t)$ and considering the time variation of the q's to be very slow, we obtain

$$G(\mathbf{k}) \simeq i \sum_{\mathbf{l},\mathbf{p}} \delta_{\mathbf{k}-\mathbf{l}-\mathbf{p}} \left[\frac{\Gamma^{\mathbf{k}}_{\mathbf{l},\mathbf{p}} a(\mathbf{l})\, a(\mathbf{p})}{\omega_l + \omega_p - \omega_k} + \frac{\Gamma^{\mathbf{k},-\mathbf{p}}_{\mathbf{l}} a(\mathbf{l}) a^*(-\mathbf{p})}{\omega_l - \omega_p - \omega_k} - \frac{\Gamma^{\mathbf{k},-\mathbf{l},-\mathbf{p}}\, a^*(-\mathbf{l}) a^*(-\mathbf{p})}{\omega_l + \omega_p + \omega_k}\right]. \tag{22}$$

Substituting (22) into (21) lets us finally write

$$T_3'(a) = \frac{i}{4}\sum_{\mathbf{l},\mathbf{p},\mathbf{n}} \delta_{\mathbf{k}+\mathbf{n}-\mathbf{l}-\mathbf{p}} C^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}} a(\mathbf{l})\, a(\mathbf{p})\, a^*(\mathbf{n}) + \text{terms not needed.} \tag{23}$$

The 'terms not needed' here have the form of the final three terms in (16). They involve rapidly oscillating exponentials and will not contribute significantly to the transport equation derived in the next section. The coefficient $C^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}}$ is given in appendix B.

3. The power spectrum of the wave amplitude

In this section we shall construct a transport equation for the spectrum of the wave amplitudes using (20). To do this we shall require ensemble averages, denoted by angular brackets, of products of the a's over many observations of the sea state.

We first observe that, since (20) is odd in the a's, it is consistent with this equation to require that all averages of the product of an odd number of a's vanish. That is,

$$\langle a(\mathbf{k})\rangle = \langle a(\mathbf{k})\, a(\mathbf{l})\, a^*(\mathbf{n})\rangle = \dots = 0.$$

We next postulate quasi-Gaussian† closure:

$$\langle a(\mathbf{l})\, a(\mathbf{p})\, a^*(\mathbf{k})\, a^*(\mathbf{n})\rangle = \langle a(\mathbf{l})\, a^*(\mathbf{k})\rangle\langle a(\mathbf{p})\, a^*(\mathbf{n})\rangle + \langle a(\mathbf{l})\, a^*(\mathbf{n})\rangle\langle a(\mathbf{p})\, a^*(\mathbf{k})\rangle. \tag{24}$$

We also assume that the averages of all other fourth-order products, such as $\langle aaaa^*\rangle$, etc., vanish.

Following Wigner (1932) we introduce the power spectrum of the a's through the definition

$$F(\mathbf{x},\mathbf{k}) = \tfrac{1}{2}\sum_{\boldsymbol{\rho}} \exp(i\boldsymbol{\rho}\,.\,\mathbf{x})\langle a(\mathbf{k}+\tfrac{1}{2}\boldsymbol{\rho})\, a^*(\mathbf{k}-\tfrac{1}{2}\boldsymbol{\rho})\rangle \tag{25a}$$

$$= (2A_0)^{-1}\int d^2r \exp(-i\mathbf{k}\,.\,\mathbf{r})\langle z(\mathbf{x}+\tfrac{1}{2}\mathbf{r})\, z^*(\mathbf{x}-\tfrac{1}{2}\mathbf{r})\rangle. \tag{25b}$$

Here

$$z(\mathbf{x},t) \equiv \sum_{\mathbf{k}} a(\mathbf{k}) \exp(i\mathbf{k}\,.\,\mathbf{x}), \tag{26}$$

which differs from the quantity (7) by terms $O(a^2)$. Using (5), we see that

$$\sum_{\mathbf{k}} F(\mathbf{x},\mathbf{k}) = \langle\zeta^2(\mathbf{x})\rangle + O(a^4). \tag{27}$$

We may therefore consider $F(\mathbf{x},\mathbf{k})$ to represent an approximation (cf. Hasselmann 1968) to the power spectrum of the wave amplitude. The precise power spectrum of the wave amplitude will contain additional terms $O(F^2)$, which can be readily evaluated.

† The closure postulate (24) has been used widely in statistical theories. An interesting argument suggesting the validity of quasi-Gaussian closure has been given by Benney & Saffman (1966) and by Benney & Newell (1969). These authors showed that when certain conditions are met the closure postulate is valid in the limit of zero modal coupling strength.

For most applications it is convenient to change from discrete to continuum normalization by replacing the sum over discrete wavenumbers by an integral through the substitution

$$\sum_{\mathbf{k}} \rightarrow \frac{A_0}{(2\pi)^2} \int d^2k. \tag{28}$$

This allows us to define

$$\Psi(\mathbf{x}, \mathbf{k}) \equiv A_0(2\pi)^{-2} F(\mathbf{x}, \mathbf{k}) \tag{29}$$

with the normalization

$$\int d^2k \Psi(\mathbf{x}, \mathbf{k}) \simeq \langle \zeta^2(\mathbf{x}) \rangle \tag{30}$$

in the approximation (27).

In practice, the Wigner spectral function is useful only if $\Psi(\mathbf{x}, \mathbf{k})$ varies slowly over distances comparable to k^{-1} for all k of interest. For oceanic applications this condition is usually well satisfied except near physical discontinuities. We thus introduce a characteristic distance W_0 over which $\Psi(\mathbf{x}, \mathbf{k})$ varies appreciably and assume that

$$k \gg W_0^{-1} \tag{31}$$

for those k of interest.† Referring back to (25) we see that (31) implies that

$$\langle a(\mathbf{k} + \tfrac{1}{2}\boldsymbol{\rho})\, a^*(\mathbf{k} - \tfrac{1}{2}\boldsymbol{\rho}) \rangle \approx 0 \tag{32}$$

for $|\boldsymbol{\rho}| \gg W_0^{-1}$.

The spectrum of the energy per unit area is, correct to second order in the a's,

$$E(\mathbf{x}, \mathbf{k}) = \rho_0 g \Psi(\mathbf{x}, \mathbf{k}),$$

where ρ_0 is the sea-water density.‡

To obtain the equation satisfied by $F(\mathbf{x}, \mathbf{k})$, we differentiate ($25a$) with respect to time (see, for example, Snider 1960):

$$\frac{\partial F(\mathbf{x}, \mathbf{k})}{\partial t} = \frac{1}{2} \sum_{\boldsymbol{\rho}} [\langle \dot{a}(\mathbf{k} + \tfrac{1}{2}\boldsymbol{\rho})\, a^*(\mathbf{k} - \tfrac{1}{2}\boldsymbol{\rho}) \rangle + \langle a(\mathbf{k} + \tfrac{1}{2}\boldsymbol{\rho})\, \dot{a}^*(\mathbf{k} - \tfrac{1}{2}\boldsymbol{\rho}) \rangle] \times \exp(i\boldsymbol{\rho} \cdot \mathbf{x}). \tag{33}$$

The time derivative of the complex amplitude $a(\mathbf{k})$ can be eliminated from (33) by substitution from (20). We then obtain on the right-hand side of (33) a sum of terms involving correlation functions such as

$$\langle a(\mathbf{k})\, a^*(\mathbf{l}) \rangle, \quad \langle a(\mathbf{p})\, a(\mathbf{l})\, a^*(\mathbf{n})\, a^*(\mathbf{k}) \rangle.$$

The latter is reduced to pair correlation functions using (24). The pair correlation functions may then be expressed in terms of F by inverting (25).

It is straightforward to evaluate the right-hand side of (33) using the above procedure and to simplify the resulting expression using the inequality (31).

† This, for example, implies that $k \gg K$ in (13).

‡ If we write $a(\mathbf{k}, t)$ to indicate the explicit time dependence of the $a(\mathbf{k})$'s, we may express the spectral distribution of the wavenumber and frequency in the form

$$\Psi(\mathbf{x}, \mathbf{k}, t, \omega) = \{A_0/[2(2\pi)^3]\} \sum_{\boldsymbol{\rho}} \int d\tau \exp(i\boldsymbol{\rho} \cdot \mathbf{x} + i\omega\tau) \times \langle a(\mathbf{k} + \tfrac{1}{2}\boldsymbol{\rho}, t + \tfrac{1}{2}\tau)\, a^*(\mathbf{k} - \tfrac{1}{2}\boldsymbol{\rho}, t - \tfrac{1}{2}\tau) \rangle.$$

The equation satisfied by this quantity is more complicated than that for $\Psi(\mathbf{x}, \mathbf{k})$ and will be described in a subsequent publication.

The F's may conveniently be replaced by Ψ's using (29). The resulting equation for Ψ is finally

$$(\partial/\partial t+\dot{\mathbf{x}}\,.\,\nabla_{\mathbf{x}}+\dot{\mathbf{k}}\,.\,\nabla_{\mathbf{k}})\,\Psi(\mathbf{x},\mathbf{k}) = \alpha(\mathbf{k})\,\Psi(\mathbf{x},\mathbf{k})+S(\mathbf{x},\mathbf{k})\,\Psi(\mathbf{x},\mathbf{k}). \tag{34}$$

In this equation

$$\dot{\mathbf{x}} = \nabla_{\mathbf{k}}\mathscr{H}, \quad \dot{\mathbf{k}} = -\nabla_{\mathbf{x}}\mathscr{H}, \tag{35}$$

where

$$\mathscr{H} = \mathbf{k}\,.\,\mathbf{U}+\omega_k-\int d^2L\, C^{\mathbf{k},\mathbf{L}}_{\mathbf{L},\mathbf{k}}\Psi(\mathbf{x},\mathbf{L}) \tag{36}$$

and

$$S(\mathbf{x},\mathbf{k}) = \{\nabla_{\mathbf{x}}\,.\,[-\mathbf{k}(\mathbf{k}\,.\,\mathbf{U})/(2k^2)+\int d^2L(\mathscr{D}_1-\mathscr{D}_2)\,\Psi(\mathbf{x},\mathbf{L})]\}. \tag{37}$$

The coefficients $\mathscr{D}_1$ and $\mathscr{D}_2$ are defined by the equations

$$\left.\begin{aligned}\mathscr{D}_1 &\equiv \nabla_{\mathbf{k}} C^{\mathbf{k},\mathbf{L}}_{\mathbf{L},\mathbf{k}},\\ \mathbf{q}\,.\,\mathscr{D}_2 &\equiv \lim_{\mathbf{q}\to 0}\,[C^{\mathbf{k},\mathbf{L}+\frac{1}{2}\mathbf{q}}_{\mathbf{L}-\frac{1}{2}\mathbf{q},\mathbf{k}+\mathbf{q}} - C^{\mathbf{k},\mathbf{L}-\frac{1}{2}\mathbf{q}}_{\mathbf{L}+\frac{1}{2}\mathbf{q},\mathbf{k}-\mathbf{q}}].\end{aligned}\right\} \tag{38}$$

We emphasize that the gradient operator $\nabla_{\mathbf{x}}$ in (37) does not act outside the curly brackets; i.e. does not act on $\Psi(\mathbf{x},\mathbf{k})$ in (34).

Equations (35) and (36) have the form of the familiar ray equations of wave propagation in the approximation of geometric optics. With $\mathscr{H} = \mathbf{k}\,.\,\mathbf{U}+\omega_k$ they have previously been used (Whitham 1961; Kenyon 1971) to study wave refraction by ocean currents. The integral term in (36) represents the influence of nonlinear wave interactions on refraction and propagation. We shall describe some implications of this term in the following section.

The first term, involving the prescribed current $\mathbf{U}$, in (37) represents the 'radiation stress' introduced by Longuet-Higgins & Stewart (1960, 1961). This term may be transformed away if Ψ is replaced by the flux spectrum

$$\Phi(\mathbf{x},\mathbf{k}) \equiv |\mathbf{C}_{\mathbf{k}}|\,\Psi(\mathbf{x},\mathbf{k}), \tag{39}$$

where $\mathbf{C}_{\mathbf{k}} = \nabla_{\mathbf{k}}\omega_k$ is the linear wave group velocity. Substitution into (34) leads to the equation

$$(\partial/\partial t+\dot{\mathbf{x}}\,.\,\nabla_{\mathbf{x}}+\dot{\mathbf{k}}\,.\,\nabla_{\mathbf{k}})\,\Phi(\mathbf{x},\mathbf{k}) = \alpha(\mathbf{k})\,\Phi(\mathbf{x},\mathbf{k})+\hat{S}(\mathbf{x},\mathbf{k})\,\Phi(\mathbf{x},\mathbf{k}). \tag{40}$$

Here

$$\hat{S}(\mathbf{x},\mathbf{k}) = \left\{\nabla_{\mathbf{x}}\,.\int d^2L\left[\mathscr{D}_1-\mathscr{D}_2-\frac{\mathbf{k}}{2k^2}C^{\mathbf{k},\mathbf{L}}_{\mathbf{L},\mathbf{k}}\right]\Psi(\mathbf{x},\mathbf{L})\right\}. \tag{41}$$

Had we kept interactions up to fifth order in the amplitudes in (20), we should have obtained Hasselmann's (1960) transport terms as additional terms on the right-hand side of (34). The contribution of this paper is thus to add two new terms to the transport equation described by Hasselmann (1968) in his review article. These are the two terms involving integrals over the spectrum in (36) and (37).

The additional term in (36) has a rather simple interpretation and is discussed in the next section. The integral term in (37) is more complex. It represents a coupling across the spectrum when the amplitude of a portion of the spectrum is spatially varied.

When explicit variation of the wind is to be taken into account, we may replace $\alpha(\mathbf{k})$ by a suitable coefficient $\alpha(\mathbf{x},\mathbf{k},t)$ in (34).

4. The phase velocity of a 'test wave'

The group velocity at a wavenumber $\mathbf{k}$ is obtained from (35) and (36) as

$$d\mathbf{x}/dt = \mathbf{U} + \mathbf{C}_{\mathbf{k}} - \int d^2L \mathscr{D}_1 \Psi(\mathbf{x}, \mathbf{L}) \equiv \mathbf{C}(\mathbf{k}). \tag{42}$$

The meaning of the first two terms is obvious. The third term represents the influence of the nonlinear wave interactions. Since $\Psi(\mathbf{x}, \mathbf{L})$ will in general be asymmetric owing to the influence of wind and/or obstructions, the group velocity can have a component not parallel to $\mathbf{k}$ induced by the nonlinear interactions.

Some insight can be obtained into (42) by considering a 'test wave' interacting with a spectrum of ocean waves in a uniform ocean. We imagine the test wave to be mechanically generated with identical characteristics for each of a sequence of observations. Thus we write

$$a(\mathbf{l}) = a_0(\mathbf{l}) + a'(\mathbf{k})\, \delta_{\mathbf{k}-\mathbf{l}}, \tag{43}$$

where a_0 is a random variable describing the ambient sea and a' represents the small amplitude test wave. We substitute (43) into (20), neglect the prescribed current, wind and viscosity terms, and obtain a linear equation for $a'(\mathbf{k})$. Because we have assumed a uniform ocean,

$$\langle a_0(\mathbf{l})\, a_0^*(\mathbf{p})\rangle = \delta_{\mathbf{l}-\mathbf{p}} \langle |a_0(\mathbf{l})|^2\rangle.$$

Integration with respect to time then gives us the angular frequency

$$\overline{\omega}_{\mathbf{k}} = \omega_k - \int d^2L\, C^{\mathbf{k},\mathbf{L}}_{\mathbf{L},\mathbf{k}} \Psi(\mathbf{L}), \tag{44}$$

where we have indicated no $\mathbf{x}$ dependence of Ψ. The phase velocity is $\overline{\omega}_{\mathbf{k}}/k$ and the group velocity deduced from (44) is in agreement with that in (42).

To illustrate the implications of (44) we consider the spectrum of Tyler *et al.* (1974), which is based on a representation proposed by Longuet-Higgins, Cartwright & Smith (1963). This is

$$\Psi(\mathbf{L}) \begin{cases} \simeq (0{\cdot}4 \times 10^{-2}/L^4)\,[G(\beta)/N] & \text{for} \quad k_0 < L < k_\Gamma, \\ = 0 \quad \text{for} \quad L < k_0 \quad \text{or} \quad L > k_\Gamma. \end{cases} \tag{45}$$

Here the angular variation of the spectrum is given by

$$G(\beta) = \alpha + (1-\alpha)\cos^{s(L)}(\tfrac{1}{2}\beta)$$

and

$$N = \int_{-\pi}^{\pi} G(\beta)\, d\beta.$$

In these equations α is very small ($\sim 10^{-2}$), k_0 and k_Γ are the long and short wavelength cut-offs of the spectrum respectively, and β is the angle between $\mathbf{L}$ and the wind direction. Finally, $s(L)$ is a function of the wavenumber which is near unity for short wavelengths and becomes quite large compared with unity near $L = k_0$, which causes the spectrum to be sharply peaked about the wind direction when L is near k_0.

We shall evaluate (44) for wavelengths much shorter than the cut-off, or

$$k \gg k_0. \tag{46}$$

In this case the principal contribution to the integral in (44) comes from values of L near k_0 and a simple analytic evaluation is possible.

The coefficient $C^{\mathbf{k},\mathbf{L}}_{\mathbf{L},\mathbf{k}}$ is obtained from equation (B 1) of appendix B. For $k \gg L$, this is

$$C^{\mathbf{k},\mathbf{L}}_{\mathbf{L},\mathbf{k}} \simeq -\tfrac{7}{4}LV_L\mathbf{k}\,.\,\mathbf{L}. \tag{47}$$

On evaluating the integral we find that

$$\overline{\omega}_{\mathbf{k}} \simeq \omega_k[1+1{\cdot}4\times 10^{-2}\cos\beta\ (k/k_0)^{\frac{1}{2}}], \tag{48}$$

where β is the angle between $\mathbf{k}$ and the wind direction. The group velocity obtained from (48) is

$$\nabla_{\mathbf{k}}\overline{\omega}_{\mathbf{k}} = \mathbf{C}_{\mathbf{k}}+\hat{\mathbf{W}}\,[1{\cdot}4\times 10^{-2}\cos\beta\ (g/k_0)^{\frac{1}{2}}], \tag{49}$$

where $\hat{\mathbf{W}}$ is a unit vector parallel to the wind direction.

During the series of experiments reported in Tyler *et al.* (1974), the 'shadow' of an island for receding waves was observed. At sufficiently large distances from the island this shadow is absent. There are evidently several possible causes for the filling in of the spectrum away from the island. One of these is nonlinear wave interactions, which we now discuss as an application of (34).

Let us suppose that at a position $\mathbf{x}$ waves of wavenumber $\mathbf{k}$ are in the shadow of the island. Then $\Psi(\mathbf{x},\mathbf{k})$ will be very small where effective shadowing occurs. On the other hand, we assume that $\Psi(\mathbf{x},\mathbf{L}) = \Psi(\mathbf{L})$ will not have much $\mathbf{x}$ dependence for those waves $\mathbf{L}$ which have 'missed' the island. If the shadowing angle is small, we can take (we now suppose that $\mathbf{U} = 0$ and the effects of wind and viscosity can be neglected) $S \simeq 0$ and $d\mathbf{k}/dt \simeq 0$ in (34). Equation (42) gives the group velocity $\mathbf{C}(\mathbf{k})$ with which waves of wavenumber $\mathbf{k}$ propagate into the shadow. If this were a time-dependent problem, with a sharply outlined shadow at $t = 0$, say $\Psi = \Psi_0(\mathbf{x},\mathbf{k})$, then at time t we should have

$$\Psi(\mathbf{x},\mathbf{k}) = \Psi_0(\mathbf{x}-\mathbf{C}(\mathbf{k})t,\mathbf{k}). \tag{50}$$

The expression (50) would lead us to expect a triangular shadow of half-angle

$$\theta_i \simeq 1{\cdot}4\times 10^{-2}\sin(2\beta)\,(k/k_0)^{\frac{1}{2}}. \tag{51}$$

When waves travelling parallel to the wind are shadowed by the island, then the filling in of the spectrum will be modified. Should a significant portion of the spectrum be in the island shadow, then $\Psi(\mathbf{L})$ in (42) must be appropriately modified.

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Appendix A. Derivation of equation (11)

In this appendix we show how to obtain the terms T_U, T_2 and T_3 in (11).

Let us first suppose that $U = 0$. Then Bernoulli's equation and the kinematic boundary condition at the surface are, respectively (see, for example, Phillips 1966),

$$\left.\begin{aligned}\partial\phi/\partial t+\tfrac{1}{2}(\nabla\phi)^2+g\zeta &= 0,\\ \partial\zeta/\partial t+(\nabla_s\phi).(\nabla_s\zeta) &= \partial\phi/\partial z,\end{aligned}\right\}\quad z=\zeta. \tag{A 1}$$

We re-express these equations in terms of $\phi(\mathbf{x},z,t)$ evaluated on the surface $z=\zeta(\mathbf{x},t)$; that is,

$$\phi_s(\mathbf{x},t)\equiv\phi[\mathbf{x},\zeta(\mathbf{x},t),t]. \tag{A 2}$$

Then, we define

$$W(\mathbf{x},t)\equiv[\partial\phi/\partial z]_{z=\zeta} \tag{A 3}$$

and rewrite (A 1) as

$$\partial\phi_s/\partial t+g\zeta = W\,\partial\zeta/\partial t-\tfrac{1}{2}(\nabla_s\phi_s-W\nabla_s\zeta)^2-\tfrac{1}{2}W^2, \tag{A 4a}$$

$$\partial\zeta/\partial t-W = -(\nabla_s\phi_s-W\nabla_s\zeta).\nabla_s\zeta. \tag{A 4b}$$

It remains to express W in terms of ϕ_s, which is a special and rather simple application of potential theory with a Dirichlet boundary condition (Jackson 1962). This is easily done by first expressing both ϕ_s and W as Taylor series in ζ about the plane $z=0$. Then W can be expressed in terms of ϕ_s by successive substitution. The result is

$$W=\Theta\phi_s-[\Theta(\zeta\Theta\phi_s)-\zeta\Theta^2\phi_s]+\{\Theta[\zeta\Theta(\zeta\Theta\phi_s)]-\zeta[\Theta^2(\zeta\Theta\phi_s)]\}$$
$$-\tfrac{1}{2}\{\Theta(\zeta^2\Theta^2\phi_s)-\zeta^2\Theta^3\phi_s\}. \tag{A 5}$$

The term $\partial\zeta/\partial t$ can be eliminated from (A 4a) using (A 4b), and W eliminated from both using (A 5). Finally, a first-order equation for

$$Z=-i\zeta+V_x^{-1}\phi_s \tag{A 6}$$

can be obtained by differentiation with respect to time and substituting from (A 4). The Fourier expansion [equation (7)] then gives us the terms T_2 and T_3 of (11).

To take account of the effect of the slowly varying current $\mathbf{U}$, we replace the left-hand sides of (A 4a, b) by the respective expressions

$$\left.\begin{aligned}&(\partial/\partial t+\mathbf{U}.\nabla_s)\phi_s+g\zeta,\\ &(\partial/\partial t+\mathbf{U}.\nabla_s)\zeta+\zeta\nabla_s.\mathbf{U}-W.\end{aligned}\right\} \tag{A 7}$$

The coefficients in T_2 in (15) are

$$\left.\begin{aligned}\Gamma^{\mathbf{k}}_{\mathbf{l},\mathbf{p}} &= \tfrac{1}{8}[(V_lV_p/V_k)(lp+\mathbf{l}.\mathbf{p})-V_p(kp-\mathbf{k}.\mathbf{p})-V_l(kl-\mathbf{k}.\mathbf{l})],\\ \Gamma^{\mathbf{k},\mathbf{p}}_{\mathbf{l}} &= \tfrac{1}{4}[(V_lV_p/V_k)(lp-\mathbf{l}.\mathbf{p})-V_p(kp+\mathbf{k}.\mathbf{p})+V_l(kl-\mathbf{k}.\mathbf{l})],\\ \Gamma^{\mathbf{k},\mathbf{l},\mathbf{p}} &= \tfrac{1}{8}[(V_lV_p/V_k)(lp+\mathbf{l}.\mathbf{p})+V_p(kp+\mathbf{k}.\mathbf{p})+V_l(kl+\mathbf{k}.\mathbf{l})].\end{aligned}\right\} \tag{A 8}$$

The first coefficient in (16) (the only one required in this paper) is

$$
\begin{aligned}
\Gamma^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}} = \tfrac{1}{4}[&(\omega_n-\omega_p)\,|\mathbf{p}-\mathbf{n}|\,(k-|\mathbf{k}-\mathbf{l}|)+(\omega_n-\omega_l)\,|\mathbf{l}-\mathbf{n}|\,(k-|\mathbf{k}-\mathbf{p}|)\\
&-(\omega_l+\omega_p)\,|\mathbf{l}+\mathbf{p}|\,(k-|\mathbf{k}+\mathbf{n}|)+\omega_p p(k-p)+\omega_l l(k-l)\\
&-\omega_n n(k-n)-\omega_l\mathbf{p}\,.\,\mathbf{n}-\omega_p\mathbf{l}\,.\,\mathbf{n}-2\omega_n\mathbf{l}\,.\,\mathbf{p}\\
&\qquad\qquad+(\omega_l\omega_n/\omega_k)\,k(n-|\mathbf{n}-\mathbf{p}|+l-|\mathbf{l}+\mathbf{p}|)\\
&+(\omega_p\omega_n/\omega_k)\,k(n-|\mathbf{n}-\mathbf{l}|+p-|\mathbf{p}+\mathbf{l}|)\\
&\qquad\qquad-(\omega_l\omega_p/\omega_k)\,k(p-|\mathbf{p}-\mathbf{n}|+l-|\mathbf{l}-\mathbf{n}|]. \qquad \text{(A 9)}
\end{aligned}
$$

Since the condition (31) has been used in our derivation of (34), we must restrict ourselves to wavelengths small compared with the length scale W_0. To do this, we suppose that the coefficients (A 8) and (A 9) vanish if any of their wavenumber arguments violates the condition (31).

Appendix B. The coefficients in equation (23)

For reference we quote the form of the coefficients of the a's in (23):

$$
\begin{aligned}
C^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}} = \Gamma^{\mathbf{k},\mathbf{n}}_{\mathbf{l},\mathbf{p}} + 4\Bigg[&\frac{\Gamma^{\mathbf{k}}_{\mathbf{l},\mathbf{p}-\mathbf{n}}\,\Gamma^{\mathbf{p}-\mathbf{n},\mathbf{n}}_{\mathbf{p}}}{\omega_p-\omega_n-\omega_{|\mathbf{p}-\mathbf{n}|}}+\frac{\Gamma^{\mathbf{k}}_{\mathbf{p},\mathbf{l}-\mathbf{n}}\,\Gamma^{\mathbf{l}-\mathbf{n},\mathbf{n}}_{\mathbf{l}}}{\omega_l-\omega_n-\omega_{|\mathbf{l}-\mathbf{n}|}}\\
&-\frac{1}{2}\frac{\Gamma^{\mathbf{k},\mathbf{n}-\mathbf{p}}_{\mathbf{l}}\,\Gamma^{\mathbf{n}-\mathbf{p},\mathbf{p}}_{\mathbf{n}}}{\omega_n-\omega_p-\omega_{|\mathbf{p}-\mathbf{n}|}}-\frac{1}{2}\frac{\Gamma^{\mathbf{k},\mathbf{n}-\mathbf{l}}_{\mathbf{p}}\,\Gamma^{\mathbf{n}-\mathbf{l},\mathbf{l}}_{\mathbf{n}}}{\omega_n-\omega_l-\omega_{|\mathbf{l}-\mathbf{n}|}}\\
&-\frac{\Gamma^{\mathbf{k},\mathbf{n}}_{\mathbf{l}+\mathbf{p}}\,\Gamma^{\mathbf{l}+\mathbf{p}}_{\mathbf{l},\mathbf{p}}}{\omega_{|\mathbf{p}+\mathbf{l}|}-\omega_l-\omega_p}+2\,\frac{\Gamma^{\mathbf{k},\mathbf{n},-(\mathbf{l}+\mathbf{p})}\,\Gamma^{-(\mathbf{l}+\mathbf{p}),\mathbf{l},\mathbf{p}}}{\omega_l+\omega_p+\omega_{|\mathbf{l}+\mathbf{p}|}}\Bigg]. \qquad \text{(B 1)}
\end{aligned}
$$

For the evaluation of the coefficients (40) certain of the terms in (B 1) appear to be singular, corresponding to the resonant excitation of arbitrarily long wavelengths. In accordance with the discussion following (A 9), these terms are to be dropped, corresponding to the assumed vanishing of the Γ coefficients.

For the evaluation of (47) one should note the sequence of cancellations of the terms with powers of k greater than the first.

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